CONTROL OF CERTAIN TRANSPORT PROCESSES

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The author proposes an approximate analytic solution of the problem of bringing a system described by a hyperbolic transport equation as rapidly as possible from one given state to another. The results of an experimental verification of the solution are presented.

In many practical situations the distribution of transport potentials in a solid at the end of a process is not a matter of indifference. For example, in drying certain materials it is important that the moisture distribution at the end of the process be sufficiently uniform. In this case it is desirable for the process to proceed as rapidly as possible. Mathematically this problem can be formulated as follows (for simplicity we will consider the isothermal case, since allowing for the effect of temperature does not introduce any important changes). Let there be a system described by the equation [1]

$$\frac{\partial^2 u}{\partial \tau^2}+2h\frac{\partial u}{\partial \tau}=a^2\frac{\partial^2 u}{\partial x^2},\quad 0\leqslant x\leqslant R,\ h>0,\ a^2>0, \tag{1}$$

with boundary conditions

$$\frac{\partial u(0,\ \tau)}{\partial x}=0,\qquad \frac{\partial u(R,\ \tau)}{\partial x}=\xi(\tau). \tag{2}$$

The initial state is characterized by the functions

$$u(x,\ 0)=f_1(x);\quad \frac{\partial u(x,\ 0)}{\partial \tau}=f_2(x). \tag{3}$$

The function $\xi(\tau)$ (see (2)) is regarded as the control function. The possibilities of control are assumed to be limited in the sense that at any moment of time the inequality

$$|\xi(\tau)|\leqslant M=\mathrm{const} \tag{4}$$

must be satisfied.

The problem consists in finding a function $\xi(\tau)$ satisfying the following requirements:

1. At some moment of time $\tau = \tau_*$ the condition

$$\int_0^R [u(x,\ \tau_*)-u^*(x)]^2\,dx=0 \tag{5}$$

must be satisfied, where $u^*(x)$ is some given function characterizing the final state of the system ($u^*(x)$ is assumed to be square integrable on the interval [0, R]).

2. τ_* must be minimal.

If the problem formulated admits an exact solution at some $\tau_* < \infty$, the approximate solution may be found as follows.

Using the Fourier finite integral cosine transformation

$$u_n(\tau)=\int_0^R u(x,\ \tau)\cos\frac{n\pi}{R}x\,dx,\quad n=0,\ 1,\ \ldots, \tag{6}$$

we find the corresponding transform of Eq. (1) with conditions (2)

$$\frac{d^2u_n}{d\tau^2}+2h\frac{du_n}{d\tau}+a^2\mu_n^2u_n=(-1)^n a^2\xi(\tau),\qquad n=0,\ 1,\ \ldots, \tag{7}$$

where

$$\mu_n = n\pi/R.$$

In the Fourier transform initial conditions (3) and condition (5) take the form

$$u_n(0)=f_{1,n},\quad \frac{du_n(0)}{d\tau}=f_{2,n}, \tag{8}$$

$$u_n(\tau_*)=u_n^*,\quad n=0,\ 1,\ \ldots \tag{9}$$

($f_{1,n}$, $f_{2,n}$, u_n^* were obtained by applying transformation (6) to the functions $f_1(x)$, $f_2(x)$, $u^*(x)$).

The initial problem reduces to the following: for system (7) with conditions (4), (8), to find a function $\xi(\tau)$ that will ensure the satisfaction of Eqs. (9) at minimal τ_*.

We will solve the analogous problem for a finite system of m + 1 equations (7), i.e., for n = 0, 1, ..., m. We denote the corresponding optimal time for this system by τ_m^*, and the optimal control by ξ_m^*. The problem of the existence and uniqueness of the solution of such a problem for finite systems of ordinary linear differential equations (in particular, of type (7)) was examined in [2].

We leave open the question of the limit $\lim\limits_{m\to\infty}\tau_m^*$, assuming that there exists a $\tau_* < \infty$, ξ_*. It is quite obvious that $\tau_m^* \leq \tau_*$, m = 0, 1, ... In fact, if we assume that at some m = k the inequality $\tau_k^* > \tau_*$ holds, then the control ξ_k^* will not be optimal for the corresponding finite problem, since there is a control ξ_* that ensures satisfaction of Eqs. (9) at $\tau_* < \tau_k^*$. From analogous reasoning there follows the validity of the inequalities $\tau_m \leq \tau_{m-1}^*$ at m = 0, 1, ... The general form of the control function ξ_m^* can be found using the maximum principle [2]. In accordance with the method and notation of [2] we form the auxiliary function H.

For this purpose we first introduce the new variable $v_n(\tau) = du_n/d_\tau$. This makes it possible to write system (7) in the form

$$\frac{dv_n}{d\tau} = -2hv_n - a^2\mu_n^2 u_n + (-1)^n a^2 \xi(\tau),$$

$$\frac{du_n}{d\tau} = v_n(\tau), \quad n = 0, 1, \ldots, m. \tag{10}$$

In accordance with [2], the function H has the form

$$H = \sum_{n=0}^{m} \{[-2hv_n - a^2\mu_n^2 u_n + +(-1)^n a^2 \xi(\tau)]\,\psi_{1,n} + v_n \psi_{2,n}\}, \tag{11}$$

where $\psi_{1,n}$, $\psi_{2,n}$ are auxiliary functions satisfying the system of equations

$$\frac{d\psi_{1,n}}{d\tau} = 2h\,\psi_{1,n} - \psi_{2,n}, \quad \frac{d\psi_{2,n}}{d\tau} = a^2\mu_n^2\,\psi_{1,n},$$

$$n = 0, 1, \ldots, m. \tag{12}$$

The unknown function ξ_m^* satisfying the conditions of the problem (at n = 0, 1, . . . , m) is found from the condition max H and consequently has the form

$$\xi_m^* = M \operatorname{sign} \sum_{n=0}^{m} (-1)^n \psi_{1n}(\tau). \tag{13}$$

Since in this case we have imposed no conditions on the values of $v_n(\tau_m^*)$, at $\tau = \tau_m^*$, the transversality conditions

$$\psi_{1,n}(\tau_m^*) = 0, \; n = 0, 1, \ldots, m$$

must be satisfied. Solving systems of equations (12) for example by means of a Laplace transformation for $\psi_{1,n}$ with allowance for the transversality conditions, we obtain

$$\psi_{1,n}(\tau) = = \frac{\psi_{1,n}(0)\exp\left[-h(\tau_m^* - \tau)\right]}{\exp(-h\tau_m^*)\sin\lambda_n\tau_m^*}\sin\lambda_n(\tau_m^* - \tau), \tag{14}$$

where

$$\lambda_n = (a^2\mu_n^2 - h^2)^{1/2}, \quad h^2 - a^2\mu_n^2 < 0,$$

$$\sin\lambda_n\tau_m^* \neq 0, \quad n = 0, 1, \ldots, m. \tag{15}$$

When $h^2 - a^2\mu_n^2 > 0$, expression (14) is also valid. Using Euler's formulas we can pass from the complex expression (14) to a real expression containing only exponential functions. Here and in what follows it is assumed that the case $h^2 - a^2\mu_n^2 = 0$ does not occur.

We will consider the following expression which gives the mean square error for control (13):

$$\delta_m^2 = \int_0^R \left[u(x, \xi_m^*, \tau_m^*) - u^*(x)\right]^2 dx. \tag{16}$$

We will show that

$$\lim_{m\to\infty} \delta_m^2 = 0, \tag{17}$$

i.e., that the initial problem can be solved with any degree of accuracy (in the sense of (16)) by choosing a sufficiently large m in the corresponding truncated problem.

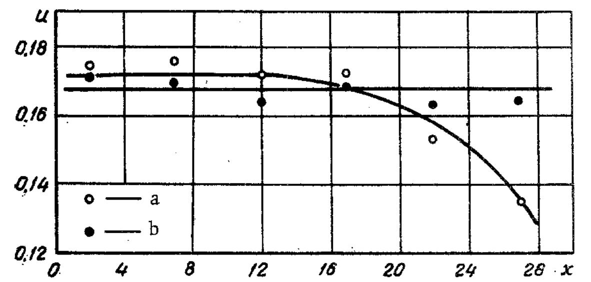

Fig. 1. Distribution of moisture content u, kg/kg, along the length of the specimen x, mm: a) at the beginning of the equalization process; b) at the end of the process; the horizontal straight line represents the calculated relation u(x,0); the curve represents $u(x, \tau_m^*)$.

The general solution of system (7) can be obtained using a Laplace transformation. It has the form

$$u_n(\tau) = \frac{1}{\alpha_{1,n} - \alpha_{2,n}} \times$$

$$\times \{[(\alpha_{1,n} + 2h) f_{1,n} + f_{2,n}] \exp\alpha_{1,n}\tau -$$

$$-[(\alpha_{2,n} + 2h) f_{1,n} + f_{2,n}] \exp\alpha_{2,n}\tau +$$

$$+(-1)^n a^2 \int_0^\tau \xi(\varepsilon)\left[\exp\alpha_{1,n}(\tau - \varepsilon) -\right.$$

$$\left.- \exp\alpha_{2,n}(\tau - \varepsilon)\right] d\varepsilon\}, \tag{18}$$

where $\alpha_{j,n} = -h + (-1)^{j+1}(h^2 - a^2\mu_n^2)^{1/2}$, j = 1, 2, $h^2 - a^2\mu_n^2 > 0$ for n = 0, 1, . . . , k. For those n > k at which $h^2 - a^2\mu_n^2 < 0$, the solution can be written in the form

$$u_n(\tau) = f_{1,n}\cos\lambda_n\tau\exp(-h\tau) +$$

$$+\frac{1}{\lambda_n}(f_{2,n} + hf_{1,n})\sin\lambda_n\tau\exp(-h\tau) +$$

$$+\frac{(-1)^n a^2}{\lambda_n}\int_0^\tau \xi(\varepsilon)\sin\lambda_n(\tau - \varepsilon) \times$$

$$\times \exp\left[-h(\tau - \varepsilon)\right] d\varepsilon,$$

$$n = k+1, \; k+2, \ldots, \tag{19}$$

where the λ_n have the form (15).

Using the formula for the inverse Fourier transformation and Eq. (9) with n = 0, 1, . . . , m, we can expand expression (16) to

$$\delta_m^2 = = \int_0^R \left\{ \frac{2}{R} \sum_{n=m+1}^{\infty} [u_n(\xi_m^*, \tau_m^*) - u_n^*] \cos \frac{n\pi}{R} x \right\}^2 dx = = \frac{4}{R^2} \sum_{n=m+1}^{\infty} [u_n(\xi_m^*, \tau_m^*) - u_n^*]^2 \int_0^R \cos^2 \frac{n\pi}{R} x dx = = \frac{2}{R} \sum_{n=m+1}^{\infty} [u_n(\xi_m^*, \tau_m^*) - u_n^*]^2. \quad (20)$$

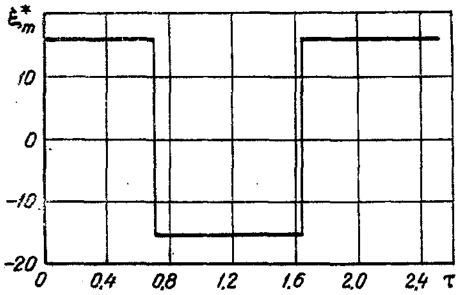

Fig. 2. Calculated relation $\xi_m^*(\tau)$ (τ in hours, ξ_m^* in kg/kg · m).

When $m \leq k$, the last sum can be divided into two parts:

$$\delta_m^2 = \frac{2}{R} \sum_{n=m+1}^{k} [u_n(\xi_m^*, \tau_m^*) - u_n^*]^2 + + \frac{2}{R} \sum_{n=k+1}^{\infty} [u_n(\xi_m^*, \tau_n^*) - u_n^*]^2, \quad (21)$$

where in the first sum the $u_n(\xi_m^*, \tau_m^*)$ correspond to solution (18) and in the second to (19).

As m increases, starting with $m = k + 1$, when the discriminant $h^2 - a^2\mu_n^2$ ($n = k + 1, k + 2, \dots$) becomes negative, expression (20) will include $u_n(\xi_m^*, \tau_m^*)$ of type (19) only.

We now obtain the following estimates:

$$|f_{1n} \cos \lambda_n \tau \exp(-h\tau) + + \frac{1}{\lambda_n} (f_{2n} + hf_{1n}) \sin \lambda_n \tau \exp(-h\tau)| < < |f_{1n}| + \frac{1}{\lambda_n} |f_{2n} + hf_{1n}| < |f_{1n}| + + \frac{1}{\lambda_{m+1}} |f_{2n} + hf_{1n}| = B_n, \quad n = m+1, m+2, \dots, \quad (22)$$

where $m > k$, and B_n is the abbreviated notation. Using (19), (22), and (4), we obtain

$$|u_n(\tau)| < B_n + + \left| \frac{a^2}{\lambda_n} \int_0^\tau \xi(\varepsilon) \sin \lambda_n (\tau - \varepsilon) \exp[-h(\tau - \varepsilon)] d\varepsilon \right| < < B_n + \left| \frac{a^2}{\lambda_n} \max_\tau |\xi| \times \times \int_0^\tau \exp[-h(\tau - \varepsilon)] d\varepsilon \right| < B_n + + \frac{a^2 M}{\lambda_n h}, \quad n = m+1, m+2, \dots, m > k. \quad (23)$$

Using (23) we find an estimate for expression (20) ($m > k$):

$$\delta_m^2 < \frac{2}{R} \sum_{n=m+1}^{\infty} \left[B_n + \frac{a^2 M}{h\lambda_n} + |u_n^*| \right]^2 < < \frac{2}{R} \sum_{n=m+1}^{\infty} (B_n + |u_n^*|)^2 + + \frac{4}{R} \sum_{n=m+1}^{\infty} (B_n + |u_n^*|) \frac{a^2 M}{h\lambda_n} + + \frac{2}{R} \sum_{n=m+1}^{\infty} \left(\frac{a^2 M}{h\lambda_n} \right)^2 \leq \frac{2}{R} \sum_{n=m+1}^{\infty} (B_n + |u_n^*|)^2 + + \frac{4}{R} \left[\sum_{n=m+1}^{\infty} (B_n + |u_n^*|)^2 \right]^{\frac{1}{2}} \left[\sum_{n=m+1}^{\infty} \left(\frac{a^2 M}{h\lambda_n} \right)^2 \right]^{\frac{1}{2}} + + \frac{2}{R} \sum_{n=m+1}^{\infty} \left(\frac{a^2 M}{h\lambda_n} \right)^2, \quad (24)$$

where the last inequality sign results from using the Cauchy inequality.

Since the functions $f_1(x)$, $f_2(x)$, $u^*(x)$ are assumed to be square integrable on the interval $[0, R]$,

$$\lim_{m \to \infty} \frac{2}{R} \sum_{n=m+1}^{\infty} (B_n + |u_n^*|)^2 = 0.$$

On the other hand, using expression (15) for λ_n, we obtain

$$\sum_{n=m+1}^{\infty} \frac{1}{\lambda_n^2} = \sum_{n=m+1}^{\infty} \frac{1}{a^2\mu_n^2 - h^2} = = \sum_{n=m+1}^{\infty} \frac{1}{a^2 - h^2/\mu_n^2} \frac{1}{\mu_n^2} < < \sum_{n=m+1}^{\infty} \frac{1}{a^2 - h^2/\mu_m^2} \frac{1}{\mu_n^2} = = \sum_{n=m+1}^{\infty} \frac{\mu_m^2 R^2}{(a^2\mu_m^2 - h^2) n^2 \pi^2} = = \frac{R^2 \mu_m^2}{\pi^2 (a^2\mu_m^2 - h^2)} \sum_{n=m+1}^{\infty} \frac{1}{n^2} = = \frac{R^2 \mu_m^2}{\pi^2 (a^2\mu_m^2 - h^2)} \left(\frac{\pi^2}{6} - \sum_{n=1}^{m} \frac{1}{n^2} \right),$$

since

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Consequently,

$$\lim_{m \to \infty} \sum_{n=m+1}^{\infty} \frac{1}{\lambda_n^2} = 0.$$

Thus, all the terms of (24) tend to zero as $m \to \infty$, and the validity of Eq. (17) has been proved. It follows that expression (13) gives the approximate solution of the initial problem. In order to find the constants $\psi_{1,n}(0)$ and τ^*_m (see (14)) that enter into the structure of the control function ξ^*_m (13), various methods have been developed. Not having the space to describe these methods in detail we refer the reader to [3]. The experimental portion of our study was concerned with the process of convective drying. The specimen was a clay cylinder 30 mm long and 30 mm in diameter, the cylindrical surface and one of the end faces being insulated. The temperature of the medium in the drying chamber was $t_{med} = 25 \pm 0.2°$ C. The air velocity in the zone where the specimen was located was 5 m/sec. The medium was moistened by atomizing distilled water with compressed air and feeding the mixture obtained into the intake zone of the drying chamber circulating fan.

The changes in moisture content were registered with six micropickups arranged at intervals of 5 mm along the length of the specimen. A detailed description of these pickups is given in [4]. The measuring error did not exceed $\pm 0.5 \cdot 10^{-2}$ kg/kg.

The initial moisture content distribution function used in the calculations is represented by the curve in Fig. 1. The final moisture distribution was required to be uniform: $u^*(x) = 0.1675 = \text{const}$ (straight line in Fig. 1). First we found the approximate values of the transport coefficients: $a^2 = 6.72 \cdot 10^{-5}$, $h = 0.45$, and then computed the approximate optimal control function (13) at $M = 15.6$, $m = 2$. Its form is shown in Fig. 2.

The values of $\frac{\partial u(x_i, 0)}{\partial \tau}$ were calculated from the approximate formula

$$\frac{\partial u(x_i, \tau_k)}{\partial \tau} \approx \frac{u(x_i, \tau_k) - u(x_i, \tau_{k-1})}{\tau_k - \tau_{k-1}}.$$

The integrals (6) for the functions $f_1(x)$, $f_2(x)$ were evaluated from the trapezoidal rule. Since it is difficult to check the variation of $\xi(\tau)$ (see (2)), we used the calculated $\xi^*_m(\tau)$ to find $u(x_*, \tau)$ at the point $x = x_* = 27 \cdot 10^{-3}$ m, i.e., at a distance of 3 mm from the exposed end face of the specimen. In finding $u(x_*, \tau)$ we used only the first three terms of the Fourier series, the solution $u(x, \tau)$ of Eq. (1). The corresponding theoretical function $u(x_*, \tau)$, converted to instrument readings, is shown in Fig. 3.

The process was controlled so that the actual (instrument readings) $u(x_*, \tau)$ varied in accordance with the curve in Fig. 3. Several such test processes were carried out. The results of one of them are represented in the figures by circles. The solid circles indicate the final moisture content distribution obtained (at $\tau^*_m = 2$ hours 42 min). The equalizing effect is obvious. This indicates the practical applicability of the approximate analytic solution of the control problem with Eq. (1).

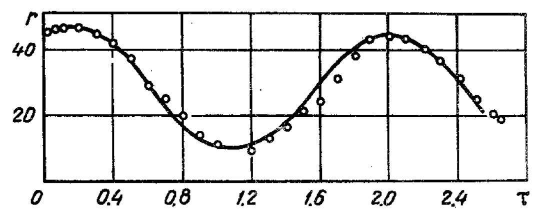

Fig. 3. Calculated (curve) and experimental (points) relation $r(\tau)$ for pickup readings at the point $x = 27$ mm (r in kilohms, τ in hours).

NOTATION

u is the specific moisture content, kg/kg; x is the space coordinate, m; τ is the time, hours; h [1/hr], a^2 [m^2/hr^2] are the transport coefficients; ξ is the control action, kg/kg · m; ψ, H are the auxiliary functions in the theory of the maximum principle; $\mu_n = n\pi/R$; $\lambda_n = (a^2\mu_n^2 - h^2)^{1/2}$; m + 1 is the number of equations (7) used in the approximate solution of the problem. Asterisks denote optimal values.

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